Technical Notes

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Quasilinearization and Optimal Control Problems with Control Bounds

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I. Introduction

In Refs. 1-3, Leondes and Paine proposed a quasilinearization method to solve optimal control problems with bounded controls without the use of a slack variable or a penalty function. This method has been demonstrated to be a powerful technique in solving practical optimal control problems with bounded controls.[4] From the computational experience of Kopp and Moyer,[5] Paine[3] noted that other ways of applying quasilinearization to handle control bounds are not straightforward. In the method of Leondes and Paine,[1-3] arbitrarily chosen initial multipliers are used to start the quasilinearization algorithm. Recently, Yeo et al.[6] suggested a logical method to choose the initial multipliers optimally. These initial multipliers are then used to start a proposed quasilinearization algorithm in Ref. 6. However, the proposed method of Refs. 1-4 and 6 cannot be applied if the optimal scalar control u cannot be expressed explicitly as a function of the state vector x, the multiplier vector λ, and the time t from Eq. (8). This paper describes a quasilinearization method that removes the above limitation without using a slack variable or a penalty function. For simplicity of illustration, a scalar control is used in this paper. The extension of the proposed method to handle problems with a control vector is straightforward. Two problems were solved to illustrate the proposed method. It should be noted that the second problem is a simple one-degree-of-freedom manipulator problem.[7]

The second problem can also be solved using two existing quasilinearization methods.[1-4,6] In the method of Yeo et al.,[6] the linearized differential equations are solved using the method of particular solutions.[9] In the method of Leondes and Paine,[1-3] they are solved using the proposed method of Ref. 8. This is the only difference between these two methods. Based on the computational experience of the author, this difference is unlikely to cause any significant difference in the computational characteristics of these two methods. Thus only the method of Yeo et al.[6] and the proposed method were used to solve the second problem in this study. The object is to compare these quasilinearization methods on the following characteristics: a) simplicity of formulation and implementation, b) computer storage requirements, c) convergence sensitivity, and d) convergence time. In addition to its ability to handle a wider class of problems, it is shown that the proposed algorithm has a larger region of convergence compared to the method of Yeo et al.[6] Other computational characteristics of these two methods are basically similar.

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II. Problem Formulation

Consider a dynamic system described by the differential equation

$$\dot{x}=f(x,u,t), \quad x(0)=x_o, \quad x(t_f)=x_f \tag{1}$$

Here the function f is an n-dimensional column vector. The symbol x, an n-dimensional column vector, denotes the state and the symbol u denotes the scalar control. The control u is constrained in the following way.

$$(u-u_{\max})+\tau(u_{\min}-u) \tag{2}$$

where $u_{\max}$ and $u_{\min}$ are upper and lower bounds of the control u. The object of the control problem is to minimize the performance index

$$J=\int_0^{t_f} L(x,u,t) \quad \mathrm{d}t \tag{3}$$

subject to the constraints of Eqs. (1) and (2), where the final time t_f is given explicitly. It is assumed that the functions L and f have continuous first and second derivatives relative to (x,u,t). It is also assumed that the optimal solution exists.

III. Necessary Conditions for Optimal Control

Define the Hamiltonian[10]

$$H=L+\lambda^T f+\sigma(u-u_{\max})+\tau(u_{\min}-u) \tag{4}$$

where

$$\sigma \begin{cases} >0, & u=u_{\max} \\ =o, & u<u_{\max} \end{cases} \tag{5}$$

$$\tau \begin{cases} >0, & u=u_{\min} \\ =0, & u>u_{\min} \end{cases} \tag{6}$$

In Eqs. (4-6), the multiplier λ is an n-dimensional column vector, σ and τ are scaler multipliers.

The Euler-Lagrange equation is

$$\dot{\lambda}=-H_x=-L_x-f_x\lambda \tag{7}$$

At a point on the optimal trajectory where $u_{\min}<u<u_{\max}$, the condition determining $u(t)$ is

$$H_u=L_u+f_u\lambda=0 \tag{8}$$

At a point on the optimal trajectory where

$$u=u_{\max} \tag{9}$$

Eq. (9) and

$$H_u=L_u+f_u\lambda+\sigma=0 \tag{10}$$

together determine u and σ. From Eq. (10),

$$\sigma=-L_u-f_u\lambda \tag{11}$$

At a point on the optimal trajectory where

$$u=u_{\min} \tag{12}$$

Eq. (12) and

$$H_u=L_u+f_u\lambda-\tau=0 \tag{13}$$

together determine u and τ. From Eq. (13),

$$\tau=L_u+f_u\lambda \tag{14}$$

IV. Quasilinearization Algorithm

Equations (1), (7), and (8) are the necessary conditions for optimal control of the formulated problem at points along the unconstrained arcs of the optimal trajectory. Suppose that at the Nth stage of the iteration, an approximate solution reasonably close to the exact solution, has been obtained. The linearized forms of Eqs. (1), (7) and (8) are

$$\dot{x}_{N+1}=\frac{\partial f}{\partial x}(x_N,u_N,t)\,[x_{N+1}-x_N]+\frac{\partial f}{\partial u}(x_N,u_N,t) \times [u_{N+1}-u_N]+f(x_N,u_N,t) \tag{15}$$

$$\dot{\lambda}_{N+1}=-\frac{\partial H_x}{\partial x}(x_N,\lambda_N,u_N,t)\,[x_{N+1}-x_N] -f_x(x_N,u_N,t)\,[\lambda_{N+1}-\lambda_N] -\frac{\partial H_x}{\partial u}(x_N,\lambda_N,u_N,t)\,[u_{N+1}-u_N]-H_x(x_N,\lambda_N,u_N,t) \tag{16}$$

and

$$\frac{\partial H_u}{\partial x}(x_N,\lambda_N,u_N,t)\,[x_{N+1}-x_N]+f_u(x_N,u_N,t) [\lambda_{N+1}-\lambda_N]+\frac{\partial H_u}{\partial u}(x_N,\lambda_N,u_N,t)\,[u_{N+1}-u_N] +H_u(x_N,\lambda_N,u_N,t)=0 \tag{17}$$

Similarly, the linearized forms of Eqs. (1),(7),(9), and 10, which are the necessary conditions for optimal control on the control bound of the optimal trajectory, are

$$\dot{x}_{N+1}=\frac{\partial f}{\partial x}(x_N,u,t)\,[x_{N+1}-x_N]+f(x_N,u,t) \tag{18}$$

$$\dot{\lambda}_{N+1}=-\frac{\partial H_x}{\partial x}(x_N,\lambda_N,u,t)\,[x_{N+1}-x_N] -f_x(x_N,u,t)\,[\lambda_{N+1}-\lambda_N]-H_x(x_N,\lambda_N,u,t) \tag{19}$$

where

$$u=u_{\max}(\text{or } u_{\min}) \tag{20}$$

and

$$u_{N+1}=u_{\max}(\text{or } u_{\min}) \tag{21}$$

After defining a new $2n$-dimensional vector

$$Y=\begin{bmatrix}x\\ \lambda\end{bmatrix} \tag{22}$$

a relation of the form

$$u_{N+1}=A(x_N,\lambda_N,u,t)\,Y_{N+1}+B(x_N,\lambda_N,u,t) \tag{23}$$

is derived from Eqs. (17) and (21). In Eq. (23), A is a $1\times 2n$ matrix and b is a 1×1 matrix. At a point on the nominal trajectory where

$$u_N\geq u_{\max} \tag{24}$$

and

$$\sigma=-L_u(x_N,u_{\max},t)-f_u(x_N,u_{\max},t)\lambda>0 \tag{25}$$

$$A=[0,\ \ldots,\ o] \tag{26}$$

$$B=[u_{\max}] \tag{27}$$

At a point on the nominal trajectory where

$$u_N\leq u_{\min} \tag{28}$$

and

$$\tau=L_u(x_N,u_{\min},t)+f_u(x_N,u_{\min},t)\lambda>0 \tag{29}$$

$$A=[0,\ \ldots,\ 0] \tag{30}$$

$$B=[u_{\min}] \tag{31}$$

If neither Eqs. (24) and (25) nor Eqs. (28) and (29) are satisfied, A and B are evaluated from Eq. (17) and $u=u_N$ in Eq. (23). After eliminating u_{N+1} from Eqs. (15) and (16) using Eq. (17), a system of linear differential equations of the form

$$\dot{Y}_{N+1}=C(x_N,\lambda_N,u,t)\,Y_{N+1}+D(x_N,\lambda_N,u,t) \tag{32}$$

is derived from Eqs. (15), (16), (18), and (19). In Eq. (32), $u=u_{\max}$ if the conditions (24) and (25) are satisfied, and $u=u_{\min}$ if the conditions (28) and (29) are satisfied. Otherwise, u is set equal to u_N. In Eq. (32), the expressions of C and D derived from Eqs. (15), (16) and (17) are different from those derived from Eqs. (18) and (19).

A step-by-step description of the quasilinearization algorithm is presented following.

a) Assume initial nominal functions $x(t)$, $u(t)$, and $\lambda(t)$. If possible, these should be consistent with the boundary conditions in Eq. (1). Alternatively, the initial function $\lambda(t)$ is selected optimally using the method of Yeo et al.[6] from given initial $x(t)$ and $u(t)$.

b) At the $N+1^{st}$ stage of the iteration ($N=0,1,2,\ldots$), the conditions (24), (25), (28), and (29) are used to determine whether a point is on the control bound along the nominal trajectory.

c) The linear differential system (32) subject to the boundary conditions in Eq. (1) is solved using the method of particular solutions.[9] Obtain the functions x_{N+1} and λ_{N+1}. Compute u_{N+1} from Eqs. (23).

d) Compute

$$\rho=\sum_{i=1}^{n}\max_{t\in[0,t_f]}\left|x_{i(N+1)}(t)-x_{i(N)}(t)\right| +\sum_{i=1}^{n}\max_{t\in[0,t_f]}\left|\lambda_{i(N+1)}(t)-\lambda_{i(N)}(t)\right| +\max_{t\in[0,t_f]}\left|u_{(N+1)}(t)-u_{(N)}(t)\right| \tag{33}$$

e) The optimal solution is obtained when $\rho<\delta$, a small preselected positive quantity. Otherwise, go to step (b) and the iterated solution is used as the nominal function for the next stage of the iteration ($N=N+1$).

V. Numerical Examples

Two numerical examples were solved using a digital computer and double-precision arithmetic. The interval of in-

tegration was divided into 100 steps and a fourth-order Runge-Kutta procedure was used to integrate first-order linear differential systems.

Convergence was defined as follows:

$$\rho < 10^{-6} \tag{34}$$

Example 1

Choose $u(t)$ in the interval $0 \le t \le t_f$ to minimize

$$J = \int_0^{t_f} [x_1^2 + x_2^2 + u^4 + u^2]\,dt \tag{35}$$

subject to

$$\dot{x}_1 = x_2, \qquad x_1(0) = 1, x_1(t_f) = 0 \tag{36}$$

$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 + u, \quad x_2(0) = 0, x_2(t_f) = 0 \tag{37}$$

and

$$-0.25 \le u \le 1 \tag{38}$$

where $t_f = 2.4$

The initial state and control functions are chosen as follows:

$$x_1(t) = 1 - t/t_f, \qquad x_2(t) = 0 \tag{39}$$

$$u(t) = \begin{cases} -0.25, & 0 \le t \le t_f/4 \\ -0.25 + 2.5(t - 0.25t_f)/t_f, & t_f/4 < t < 3t_f/4 \\ 1, & 3t_f/4 \le t \le t_f \end{cases}$$

The optimal initial multipliers $\lambda_1(t)$, $\lambda_2(t)$ were then generated using the method of Ref. 6. The assumptions in Eq. (39) result in the choice of the matrices

$$S_\theta = [\ell_1] \tag{40}$$

where

$$\ell_1 = \begin{cases} 0, & 0 \le t \le t_f/4 \\ 1, & t_f/4 < t < 3t_f/4 \\ 0, & 3t_f/4 \le t \le t_f \end{cases} \tag{41}$$

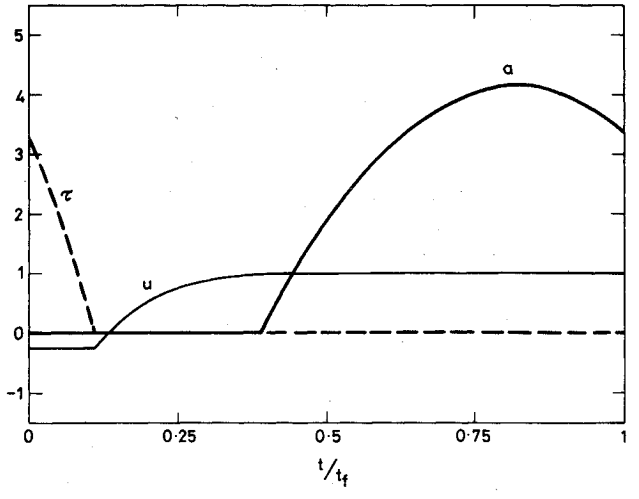

Fig. 1 Converged functions $u(t)$, $\sigma(t)$ and $\tau(t)$ of Example 1.

Table 1 Numbers of iterations at convergence N_1^*, N_2^* vs the parameter k

k	N_1^*	N_2^*
-10^6	Divergence	7
-10^4	Divergence	7
-10^2	Divergence	7
-1	Divergence	7
-0.25	Divergence	7
-0.20	Divergence	7
-0.15	Divergence	7
-0.10	6	7
-0.05	6	7
0	6	7
0.05	6	7
0.10	6	7
0.15	6	7
0.20	Divergence	7
0.25	Divergence	7
1	Divergence	7
10^2	Divergence	7
10^4	Divergence	7
10^6	Divergence	7

and

$$S_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{42}$$

in the auxiliary minimization problem of Ref. 6. Together with initial state and control functions in Eq. (39), the optimally selected multipliers $\lambda_1(t)$, $\lambda_2(t)$ were used to start the proposed quasilinearization algorithm, the solution converges to the desired accuracy in nine iterations. The converged functions $u(t), \sigma(t)$ and $\tau(t)$ are displayed in Fig. 1.

Example 2

Choose $u(t)$ in the interval $0 \le t \le t_f$ to minimize

$$J = \tfrac{1}{2}\int_0^{t_f} u^2\,dt \tag{43}$$

Subject to the constraints

$$\dot{x}_1 = x_2, x_1(0) = 0, \quad x_1(t_f) = 1 \text{ rad} \tag{44}$$

$$\dot{x}_2 = 8.55u - 2.406x_2, \quad x_2(0) = 0, x_2(t_f) = 0$$

and

$$-1 \le u \le 1 \tag{45}$$

where $t_f = 0.75$ sec.

The following initial state and control variables were assumed.

$$x_1(t) = t/t_f, \qquad x_2(t) = 0 \tag{46}$$

$$u(t) = \begin{cases} 1, & 0 \le t \le 0.1t_f \\ 1 - 2(t - 0.1t_f)/0.8t_f, & 0.1t_f < t < 0.9t_f \\ -1, & 0.9t_f \le t \le t_f \end{cases}$$

Computer runs were made by choosing the initial multipliers arbitrarily, specifically in the form

$$\lambda_1(t) = k, \qquad \lambda_2(t) = kt/t_f \tag{47}$$

where k is a parameter in the range

$$-10^6 \le k \le 10^6 \tag{48}$$

Table 1 shows the numbers of iterations at convergence N_1^*, N_2^* vs the parameter k, where N_1^* is the number of iterations at convergence using the method of Yeo et al.,[6] N_2^* is the number of iterations using the proposed method. Obviously, the proposed method has much larger region of convergence.

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Thermal Decomposition Studies on Ammonium Perchlorate-Based Composite Propellants

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Introduction

THE role of condensed-phase reactions in solid propellant combustion has long been debated. The work of Waesche and Wenograd,[1] and Sammons[2] in the late seventies demonstrated the importance of such reactions in the overall combustion process of a solid-propellant. Our recent studies on polystyrene (PS), ammonium perchlorate (AP) propellant system have also shed some light on the occurrence of condensed-phase reactions.[3,4] We have shown further that the thermal decomposition of the oxidizer and the propellant in this system is correlated to its burning rate.[5] In order to have a deeper insight into the thermal decomposition process, a quantitative analysis of the thermal changes associated with the binder, oxidizer, and the propellant is necessary. The enthalpy and kinetic estimations from the Differential Scanning Calorimetric (DSC) thermograms of PS, AP, and PS/AP propellant form the primary objective of the present work. Another objective is to compare the activation energies (E) of the thermal decomposition of PS and AP to that of the propellant.

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Experimental

PS and PS/AP propellants were prepared as described earlier.[6] Powdered AP from Fischer Scientific Co. was used without any further purification. DSC thermograms were obtained by one of the authors (K. Kishore) at the University of Leeds, Great Britain. Isothermal and scanning thermograms were obtained on Perkin-Elmer DSC-1B Differential Scanning Calorimeter. The details of the operation of the instrument, generation of α (fraction decomposed) vs time plot, and the calculation of E from the thermograms have been described in detail in our recent publications.[7]

Results and Discussion

The DSC thermograms obtained in scanning and isothermal operations have been presented in Figs. 1 and 2. The enthalpy data for AP, PS, and AP/PS propellants obtained from the analysis of DSC thermograms are presented in Tables 1 and 2. AP endotherm starts appearing at 517 ± 2 K. The endothermic enthalpy change was calculated to be 2.39 ± 0.07 Kcal/mole which agrees well with other reported literature values.[7]

The enthalpies, for different scanning thermograms, of both low temperature exotherm (LTE) and high temperature exotherm (HTE) are given in Table 1. In order to confirm the validity of these values, isothermal runs were also carried out for LTE and the enthalpy thus calculated for isothermal runs agrees well with that from scanning runs.[7] Assuming that the nature of decomposition is the same in LTE and HTE, we observed that 30% decomposition occurs in LTE (scanning and isotherming both operations) which is in accordance with the reported values from weight loss measurements. The total exothermic heat release of both LTE and HTE was estimated to be 278 ± 10 cal/g, which is in agreement with the earlier reported value (see Table 1) by Wenograd[1] et al.

The E values for LTE and HTE in scanning operation were calculated by the method described earlier[7] and found to be 22.5 (21.5 in isothermal operation) and 60.5 Kcal/mole, respectively, which agree fairly well with other values using other techniques. This clearly indicates the soundness of the method used for kinetic analysis from DSC thermograms.

The propellant endotherm, appearing at 523 ± 5 K, was subjected to enthalpy estimations and was found to correspond quantitatively to the AP phase-transition endotherm (Table 1). This suggests that AP phase transformation is not affected during the casting of the propellant.

The propellant exotherm starts appearing (Fig. 1) at a later stage than the first exotherm of AP. This can be explained on the basis of a physical model[4] which considers that AP particle or particles are surrounded by the binder. The polymer shells containing AP are closely packed in the matrix so that they make the matrix porous. According to this model, the decomposition of AP starts inside the polymer film, which further undergoes heterogeneous and homogeneous reactions with the decomposition products arising from the degradation of the inner layer of the film. This continues until finally the film is broken because of polymer degradation in its entirety, or because of build-up of the gas pressure. Polymer, being a bad thermal conductor, transfers the heat to the pan with some time lag, and that it why the actual temperature recording is shifted ahead. This also explains why the propellant exotherm finishes earlier compared to PS and AP (Fig. 1).

Since the enthalpy changes for PS, AP, and propellant (Table 1) are known, one can calculate the heat generated by the condensed-phase reactions, provided it is assumed that DSC records only solid-state reactions especially when the